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The Maximal Number of Orbits of a Permutation Group with Bounded Movement*

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1. INTRODUCTION

Let G be a permutation group on a set Ω such that G has no fixed points in Ω. Then G is said to have *bounded movement* if, for some positive integer m and for all $g \in G$ and $\Gamma \subseteq \Omega$, the cardinality $|\Gamma^g \setminus \Gamma|$ is at most m. When the maximum of $|\Gamma^g \setminus \Gamma|$ over all $g \in G$ and $\Gamma \subseteq \Omega$ is equal to m, we say G has *bounded movement equal to m*.

The third author has investigated groups with bounded movement in [6], using a fundamental result of B. H. Neumann [3] which was transformed by P. M. Neumann [4, Lemma 2.3] into a theorem about the separation of subsets of points under a group action (see [5] for a survey). For a group G of bounded movement equal to m, and having no orbits of length 1, it was

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proved in [6] that both the number of G-orbits and the length of each G-orbit are finite, with upper bounds which are linear in m. In particular it was shown there that the number of G-orbits is at most $2m - 1$. We present here a classification of the groups for which the bound $2m - 1$ is attained (see [5, Question 3]).

We shall say that an orbit of a permutation group is *nontrivial* if its length is greater than 1. The groups described below are examples of permutation groups of bounded movement equal to m having exactly $2m - 1$ orbits, all of them nontrivial.

EXAMPLE. Let $m = 2^{r-1} \geq 1$, and let $G := Z_2^r$. Then G has $2^r - 1 = 2m - 1$ subgroups of index 2, say $H_1, \ldots, H_{2m-1}$. For $i = 1, \ldots, 2m - 1$, let Ω_i denote the set of two cosets of H_i in G, and set $\Omega := \bigcup_{i=1}^{2m-1} \Omega_i$. Then G acts faithfully on Ω by right multiplication with $2m - 1$ orbits $\Omega_1, \ldots, \Omega_{2m-1}$, each of length 2. Each nontrivial element $g \in G$ lies in exactly $2^{r-1} - 1 = m - 1$ of the subgroups H_i and permutes nontrivially the remaining $m = 2^{r-1}$ of the Ω_i. Thus each nontrivial element of G has $m = 2^{r-1}$ cycles of length 2 in Ω. For any subset $\Gamma \subseteq \Omega$ and any $g \in G$, the set $\Gamma^g \setminus \Gamma$ consists of at most 1 point from each of the G-orbits on which g acts nontrivially, and hence $|\Gamma^g \setminus \Gamma| \leq m$. It follows that G has bounded movement equal to m and G has $2m - 1$ nontrivial orbits in Ω.

It turns out that for $m > 1$, these are the only examples meeting the bound.

THEOREM 1. *Let m be a positive integer, and suppose that G is a permutation group on a set Ω such that G has no fixed points in Ω, G has bounded movement equal to m, and G has $2m - 1$ orbits in Ω. Then m is a power of 2, and either*

(i) $m = 1$, $|\Omega| = 3$, *and G is Z_3 or S_3, or*

(ii) *G is one of the groups in the example above, that is, G is elementary abelian of order $2m$, all G-orbits have length 2, and the pointwise stabilizers of the G-orbits are precisely the $2m - 1$ distinct subgroups of G of index 2.*

It was pointed out to us by Avinoam Mann that Theorem 1 yields, as an immediate corollary, a classification of groups satisfying another extremal condition. In [6] it was shown that, for a permutation group G on a set Ω, with no fixed points in Ω, such that G has bounded movement equal to m, the size of Ω is at most $5m - 2$. If $|\Omega| = 5m - 2$ then it follows easily from the analysis in [6] that G has $2m - 1$ orbits in Ω, and then Theorem 1 implies that $m = 1$ and G is Z_3 or S_3. Thus we have the following corollary to Theorem 1.

COROLLARY TO THEOREM 1. *Let m be a positive integer and suppose that G is a permutation group on a set Ω, with no fixed points in Ω, such that G has bounded movement equal to m. Then $|\Omega| \leq 5m - 2$ with equality if and only if $m = 1$ and $G = Z_3$ or S_3.*

The classification in Theorem 1 in the case where $m = 1$ can be deduced immediately from the classification by Leonid Brailovsky [1] of subsets of movement 1 relative to some group action. For a positive integer m, a subset $\Gamma \subseteq \Omega$ is said to have *movement m* relative to a permutation group $G \leq \mathrm{Sym}(\Omega)$ if $|\Gamma^g \setminus \Gamma|$ is bounded above, for $g \in G$, and its maximum value is m. We write move $(\Gamma) = m$. Brailovsky [1] showed that a subset of Ω has movement 1 relative to $G \leq \mathrm{Sym}(\Omega)$ if and only if it is equal to a union of G-orbits in Ω with either one point added or one point removed.

When $m > 1$ the classification in Theorem 1 follows immediately from the following theorem about subsets with movement m. This theorem was proved and shown to us by Peter Neumann after seeing a draft of our paper which had been submitted for publication. On the suggestion of the editors and with Peter's agreement we rewrote our paper incorporating his result. Our exposition uses the language of permutation groups, whereas Peter's proof was written in terms of covers for abstract groups.

Let G be a permutation group on a set Ω with orbits Ω_i, for $i \in I$. We shall say that a subset $\Gamma \subseteq \Omega$ *cuts across* each G-orbit if

$$\Gamma_i := \Gamma \cap \Omega_i \notin \{\emptyset, \Omega_i\}$$

for all $i \in I$.

THEOREM 2. *Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group with t orbits. Suppose that $\Gamma \subseteq \Omega$ is a set such that $\mathrm{move}(\Gamma) = m > 1$, and such that Γ cuts across each G-orbit. Then $t \leq 2m - 1$, and moreover if $t = 2m - 1$ then*

(1) *G is an elementary abelian 2-group and every G-orbit has size 2;*

(2) *if the rank of G is r then $r \geq 2$, $t = 2^r - 1$, and $m = 2^{r-1}$;*

(3) *the t different G-orbits are (isomorphic to) the coset spaces of the $2^r - 1$ different subgroups of index 2 in G.*

The assertion that $t \leq 2m - 1$ was proved in [2, Lemma 3.5]. It is the characterization of the groups for which the upper bound is attained which concerns us here. From our observations above, all of our results will follow from this characterization. Thus the rest of the paper is devoted to proving it.

2. PROOF OF THEOREM 2

Let m be a positive integer greater than 1. Suppose that $G \leq \mathrm{Sym}(\Omega)$ with orbits $\Omega_1, \Omega_2, \dots, \Omega_t$, where $t = 2m - 1$. Suppose further that $\Gamma \subseteq \Omega$ has $\mathrm{move}(\Gamma) = m$ and that Γ cuts across each of the G-orbits Ω_i. For each i set $n_i := |\Omega_i|$ and $\Gamma_i := \Gamma \cap \Omega_i$. Note that $0 < |\Gamma_i| < n_i$.

Claim. If Theorem 2 holds for the special case in which $|\Gamma_i| = 1$ for each $i = 1, \dots, 2m - 1$, then it holds in general.

Suppose that Theorem 2 holds for the case where each $|\Gamma_i| = 1$. For $i = 1, \dots, t$, define $\Sigma_i := \{\Gamma_i^g | g \in G\}$, and note that $|\Sigma_i| \geq 2$ since Γ cuts across Ω_i. Set $\Sigma := \bigcup_{i \geq 1} \Sigma_i$. Then G induces a natural action on Σ for which the G-orbits are $\Sigma_1, \dots, \Sigma_t$. Let G^{Σ} denote the permutation group induced by G on Σ, and let K denote the kernel of this action.

We claim that the t-element subset $\Gamma_{\Sigma} = \{\Gamma_1, \dots \Gamma_t\} \subseteq \Sigma$ has movement equal to m relative to G^{Σ}, and that Γ_{Σ} cuts across each G^{Σ}-orbit Σ_i. For each $g \in G$, $|\Gamma^g \setminus \Gamma| \leq m$ and hence $|\Gamma_{\Sigma}^g \setminus \Gamma_{\Sigma}| \leq m$. Thus $\mathrm{move}(\Gamma_{\Sigma}) \leq m$. Also, since $|\Sigma_i| \geq 2$ and $\Gamma_{\Sigma} \cap \Sigma_i$ consists of the single element Γ_i of Σ_i, the set Γ_{Σ} cuts across each of the $2m - 1$ orbits Σ_i. However, it follows from [2, Lemma 3.5] that the number of G^{Σ}-orbits is at most $2 \cdot \mathrm{move}(\Gamma_{\Sigma}) - 1$, and hence $\mathrm{move}(\Gamma_{\Sigma}) = m$.

Thus the hypotheses of Theorem 2 hold for the subset $\Gamma_{\Sigma} \subseteq \Sigma$ relative to G^{Σ}, and Γ_{Σ} meets each G^{Σ}-orbit in exactly one point. By our assumption it follows that $t = 2^r - 1 = 2m - 1$ for some $r > 1$, and that $G^{\Sigma} = Z_2^r$ and each $|\Sigma_i| = 2$. Further, the subgroups H_i of G fixing Γ_i setwise range over the $2^r - 1$ distinct subgroups which have index 2 in G and which contain K. In particular, for each i, H_i is normal in G and hence the H_i-orbits in Ω_i are blocks of imprimitivity for G, and their number is at most $|G : H_i| = 2$. Since H_i fixes Γ_i setwise it follows that Γ_i is an H_i-orbit and $n_i = 2|\Gamma_i|$.

Let $g \in G \setminus K$. Then in its action on Σ, g moves exactly m of the Γ_i. Since the Γ_i are blocks of imprimitivity for G, each Γ_i^g is equal to either Γ_i or $\Omega_i \setminus \Gamma_i$. It follows that $|\Gamma^g \setminus \Gamma|$ is equal to the sum of the sizes of the m subsets Γ_i moved by g. However, since $\mathrm{move}(\Gamma) = m$, each of these m subsets Γ_i must have size 1. Since for each i we may choose an element g which moves Γ_i, we deduce that each of the Γ_i has size 1, and that K is the identity subgroup. It follows that Theorem 2 holds for G. Thus the claim is proved.

From now on we may and shall assume that each $|\Gamma_i| = 1$. Let $\Gamma_i = \{\omega_i\}$. Further we may assume that $n_1 \leq n_2 \leq \cdots \leq n_t$. For $g \in G$ let $c(g)$ denote the number of integers i such that $\omega_i^g = \omega_i$. Note that since $\mathrm{move}(\Gamma) = m$ we have $c(g) \geq t - m = m - 1$, and also $c(1_G) = t > m - 1$. Next we show that at least one of the Ω_i has length 2.

LEMMA 2.1. $n_1 = 2$.

Proof. Let X denote the number of pairs (g, i) such that $g \in G$, $1 \le i \le t$, and $\omega_i^g = \omega_i$. Then $X = \sum_{g \in G} c(g)$, and by our observations, $X > |G| \cdot (m-1)$. On the other hand, for each i, the number of elements of G which fix ω_i is $|G_{\omega_i}| = |G|/n_i$, and hence $X = |G| \sum_{i=1}^t n_i^{-1}$. If all the $n_i \ge 3$, then $X \le |G| \cdot t/3 \le |G| \cdot (m-1)$ (since $m \ge 2$) which is a contradiction. Hence $n_1 = 2$. ∎

A similar argument to this enables us to show that all the $n_i = 2$, and hence that G is an elementary abelian 2-group.

LEMMA 2.2. *The group $G = Z_2^r$ for some $r \ge 2$. Moreover each $n_i = 2$, the stabilizers G_{ω_i} $(2 \le i \le t)$ are pairwise distinct subgroups of index 2 in G, and for each $g \ne 1$, $c(g) = m - 1$.*

Proof. By Lemma 2.1, $n_1 = 2$. Thus $H := G_{\omega_1}$ is a subgroup of index 2. This time we compute the number Y of pairs (g, i) such that $g \in G \setminus H$, $2 \le i \le t$, and $\omega_i^g = \omega_i$. For each such g, $\omega_1^g \ne \omega_1$ and hence there are $c(g)$ of these pairs with first entry g. Thus

$$Y = \sum_{g \in G \setminus H} c(g) \ge |G \setminus H| \cdot (m-1) = |G| \cdot \frac{m-1}{2}.$$

On the other hand, for each $i \ge 2$, the number of elements of $G \setminus H$ which fix ω_i is $|G_{\omega_i} \setminus H|$. If $G_{\omega_i} = H$ then $|G_{\omega_i} \setminus H| = 0$, while if $G_{\omega_i} \ne H$ then $|G_{\omega_i} \setminus H| = |G_{\omega_i}|/2 = |G|/2n_i \le |G|/4$. Hence

$$Y = \sum_{i=2}^{t} |G_{\omega_i} \setminus H| \le \frac{|G|}{2} \sum_{i=2}^{t} \frac{1}{n_i} \le \frac{|G|}{4} \cdot (t-1) = |G| \cdot \frac{m-1}{2}.$$

It follows that equality holds in both of the displayed approximations for Y. This means in particular that each $n_i = 2$, whence $G = Z_2^r$ for some r. Further, for each $i \ge 2$, $G_{\omega_i} \ne H$ and so $r \ge 2$. Arguing in the same way with H replaced by G_{ω_i}, for some $i \ge 2$, we see that $G_{\omega_i} \ne G_{\omega_j}$ if $j \ne i$, and also if $g \notin G_{\omega_i}$ then $c(g) = m - 1$. Thus the stabilizers G_{ω_i} $(1 \le i \le t)$ are pairwise distinct, and if $g \ne 1$ then $c(g) = m - 1$. ∎

Finally we determine m.

LEMMA 2.3. $m = 2^{r-1}$.

Proof. We use the information in Lemma 2.2 to determine precisely the quantity $X = \sum_{g \in G} c(g)$:

$$X = t + (|G| - 1) \cdot (m-1) = 2m - 1 + (2^r - 1)(m-1).$$

On the other hand, from the proof of Lemma 2.1,

$$X = |G| \sum_{i=1}^{t} n_i^{-1} = \frac{|G| \cdot t}{2} = 2^{r-1} \cdot (2m - 1).$$

Thus $(2^{r-1} - 1)(2m - 1) = (2^r - 1)(m - 1)$. Since the integers $2^r - 1$ and $2^{r-1} - 1$ are relatively prime, $2^r - 1$ divides $2m - 1$, and since the integers $2m - 1$ and $m - 1$ are relatively prime, $2m - 1$ divides $2^r - 1$. It follows that $2m - 1 = 2^r - 1$, whence $m = 2^{r-1}$. ∎

The proof of Theorem 2 now follows from Lemmas 2.1–2.3.

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